

Master equation for spin-spin correlation functions of the XXZ chain

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Abstract

We derive a new representation for spin-spin correlation functions of the finite XXZ spin-1/2 Heisenberg chain in terms of a single multiple integral, that we call *the master equation*. Evaluation of this master equation gives rise on the one hand to the previously obtained multiple integral formulas for the spin-spin correlation functions and on the other hand to their expansion in terms of the form factors of the local spin operators. Hence, it provides a direct analytic link between these two representations of the correlation functions and a complete re-summation of the corresponding series. The master equation method also allows one to obtain multiple integral representations for dynamical correlation functions.

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1 Introduction

One of the central question in the theory of quantum integrable models [1, 2, 3, 4] is the exact computation of their correlation functions. Apart from few cases, like free fermions [5, 6, 7, 8, 9, 10] or conformal field theories [11], this problem is still far from its complete solution. In particular, the computation of manageable expressions for two point functions of local operators and their asymptotic behavior at large distance is a central open problem. If one considers the case $T = 0$, such a problem reduces to the calculation of the average value in the ground state $|\omega\rangle$ of the product of two local operator θ_1, θ_2 :

$$g_{12} = \langle \omega | \theta_1 \theta_2 | \omega \rangle. \quad (1.1)$$

There are basically two main strategies to evaluate such a function:

(i) to compute the action of local operators on the ground state $\theta_1 \theta_2 |\omega\rangle = |\tilde{\omega}\rangle$ and then to calculate the resulting scalar product $g_{12} = \langle \omega | \tilde{\omega} \rangle$;

(ii) to insert a sum over a complete set of states $|\omega_i\rangle$ (for instance, a complete set of eigenstates of the Hamiltonian) between the local operators θ_1 and θ_2 and to obtain the representation for the correlation function as a sum over one-point matrix elements (form factor type expansion [12, 13, 14])

$$g_{12} = \sum_i \langle \omega | \theta_1 | \omega_i \rangle \cdot \langle \omega_i | \theta_2 | \omega \rangle. \quad (1.2)$$

The aim of this paper is to give a direct, analytic relation between the approaches (i) and (ii) in the case of the XXZ spin- $\frac{1}{2}$ finite chain. For the sake of simplicity we mainly consider the case of zero-temperature and equal-time σ^z correlation function. It will be clear however that our method is more general. In particular, it allows one to compute dynamical correlation functions. The corresponding results will be described in a sequel to the present paper.

We consider the periodical XXZ spin- $\frac{1}{2}$ Heisenberg chain in an external magnetic field. The Hamiltonian of this model is given by [15]

$$H = \sum_{m=1}^M (\sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + \Delta (\sigma_m^z \sigma_{m+1}^z - 1)) - h S_z, \quad (1.3)$$

where

$$S_z = \frac{1}{2} \sum_{m=1}^M \sigma_m^z, \quad [H, S_z] = 0. \quad (1.4)$$

Here Δ is the anisotropy parameter, h the external classical magnetic field, $\sigma_m^{x,y,z}$ are the spin operators (in the spin- $\frac{1}{2}$ representation), associated with each site of the chain m , and M is even. The quantum space of states is $\mathcal{H} = \otimes_{m=1}^M \mathcal{H}_m$, where $\mathcal{H}_m \sim \mathbb{C}^2$ is called the local quantum space at site m . The operators $\sigma_m^{x,y,z}$ act as the corresponding Pauli matrices in the space \mathcal{H}_m and as the identity operator elsewhere.

The method to compute eigenstates and energy levels (Bethe ansatz) of the Hamiltonian (1.3) was proposed by Bethe in 1931 in [16] and developed later in [17, 18, 19]. The algebraic version of the Bethe ansatz was created in the framework of the Quantum Inverse Scattering Method by L.D. Faddeev and his school [20, 21, 4]. However the knowledge of the correlation functions of the XXZ chain has for a long time been restricted to the free fermion point $\Delta = 0$.

In the case of the XXZ chain the approach (i) leads to multiple integral representations for the correlation functions. In the thermodynamic limit, at zero temperature and for zero magnetic field, such representations were obtained from the q -vertex operator approach (also using corner transfer matrix technique) in the massive regime $\Delta > 1$ in 1992 [22] and conjectured in 1996 [23] for the massless regime $-1 < \Delta \leq 1$ (see also [24]). A proof of these results together with their extension to non-zero magnetic field was given in 1999 [25, 26] for both regimes using algebraic Bethe ansatz and the actual resolution of the so-called quantum inverse scattering problem [25, 27]. Using these results, the spontaneous magnetization was obtained in [28]. For the case of the XXX model ($\Delta = 1$) this type of representation was later studied in [29, 30]. Integral representations for spin-spin correlation functions were derived in [31, 32]. Recently the generalization of this result for finite temperature was given in [33].

In the framework of the approach (ii) integral representations for the form factors of the XXZ chain in the thermodynamic limit were obtained in [22, 24, 34, 35, 36, 37]. Determinant representations for the form factors of the finite chain were computed in [25, 28]. An effective summation of the form factor series in the case of free fermions was done in [38, 39].

To explain our method we will study the two point correlation function of the third components of spin $\langle \sigma_1^z \sigma_{m+1}^z \rangle$. Following the papers [46, 39, 31] we use for this purpose a special generating function $\langle Q_{1,m}^\kappa \rangle$ (see (4.1), (4.2)). On the way to relate the previously obtained multiple integral representation of the spin-spin correlation function to its form factor type expansion we derive what we call *the master equation* (see (4.4)) for this generating function. It is given as a single multiple integral of Cauchy type over a certain contour Γ in \mathbb{C}^N

$$\langle Q_{1,m}^\kappa \rangle = \oint_{\Gamma} dz_1 \dots dz_N F_\kappa(\{z\}). \quad (1.5)$$

The integrand $F_\kappa(\{z\})$ is a function of the N variables z_1, \dots, z_N . It is mainly given in terms of the eigenvalues τ_κ of the twisted transfer matrix (see (3.1)) and of the functions \mathcal{Y}_κ defining the corresponding twisted Bethe equations (see (3.5), (3.6)). It is a periodical function of each argument z_j , vanishing at $z_j \rightarrow \pm\infty$. Therefore there are two ways to evaluate the integral (1.5): either to compute the residues in the poles inside Γ , or to compute the residues in the poles within strips of the width $i\pi$ outside Γ .

The first way leads to a representation of the correlation function $\langle \sigma_1^z \sigma_{m+1}^z \rangle$ in terms of the previously obtained [31] m -multiple integrals. Evaluation of the integral (1.5) in terms of the poles outside Γ gives us the form factor type expansion of the correlation function (i.e. an expansion in terms of matrix elements of σ^z between the ground state and all excited states).

We would like to stress that the objects entering the above master equation for the finite XXZ chain are quite generic in the context of quantum integrable models solvable by the algebraic Bethe ansatz. Therefore we conjecture that similar formula holds true in more general situations, in particular for the field theory models. For these models the approach (i) usually leads to short distance expansions for the correlation functions, while the approach (ii) gives their long distance expansion. The problem of putting into explicit (analytic) correspondence these two regimes has been the subject of several works along the last fifteen years (see e.g., [40, 41, 42, 43]), remaining however up to now an open problem. We hope that the method presented here could ultimately shed some new light on these topics.

The paper is organized as follows. In the next section we give the main notations and definitions of the XXZ model using the framework of the algebraic Bethe ansatz. In Section 3 the twisted transfer matrix \mathcal{T}_κ is introduced. We describe the properties of this operator and related objects, like twisted Bethe equations, their solutions and the scalar product formulas for the eigenstates of \mathcal{T}_κ with arbitrary vectors. In Section 4 we derive the master equation. The last two sections are devoted to the above mentioned two evaluations of the multiple contour integral defining this master equation. In Section 5 we reproduce from it the multiple integral representation for the σ^z correlation function obtained in [31]. This representation is given for the finite chain and in the thermodynamic limit both for the massless and massive regimes. In Section 6 we obtain the form factor type expansion for the σ^z correlation function. In the Conclusion we give several comments on the key points of our methods. We also discuss several perspectives of the obtained result and announce the multiple integral representation for the dynamical correlation function of the third components of spin, which in fact is the subject of a forthcoming publication [49].

2 Algebraic Bethe ansatz for the XXZ chain

In the framework of the algebraic Bethe ansatz the Hamiltonian (1.3) can be obtained from the monodromy matrix $T(\lambda)$, that in its turn is completely defined by the R -matrix. The R -matrix of the XXZ chain acts in the space $\mathbb{C}^2 \otimes \mathbb{C}^2$ and is equal to¹

$$R(\lambda) = \begin{pmatrix} \sinh(\lambda + \eta) & 0 & 0 & 0 \\ 0 & \sinh \lambda & \sinh \eta & 0 \\ 0 & \sinh \eta & \sinh \lambda & 0 \\ 0 & 0 & 0 & \sinh(\lambda + \eta) \end{pmatrix}, \quad \cosh \eta = \Delta. \quad (2.1)$$

It is a solution of the Yang-Baxter equation. Identifying one of the two vector spaces of the R -matrix with the quantum space \mathcal{H}_m , we obtain the quantum L -operator at the site m

$$L_m(\lambda) = R_{0m}(\lambda - \eta/2). \quad (2.2)$$

¹Note that we use here a different normalization for the R -matrix compared to [31].

Here R_{0m} acts in $\mathbb{C}^2 \otimes \mathcal{H}_m$. Then the monodromy matrix $T(\lambda)$ is constructed as an ordered product of the L -operators with respect to all the sites of the chain

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} = L_M(\lambda) \dots L_2(\lambda) L_1(\lambda). \quad (2.3)$$

The operator $T(\lambda)$ acts in $V \otimes \mathcal{H}$, where $V \sim \mathbb{C}^2$ is usually called auxiliary space of $T(\lambda)$. The Hamiltonian (1.3) at $h = 0$ is related to $T(\lambda)$ by a ‘trace identity’

$$H = 2 \sinh \eta \left. \frac{dT(\lambda)}{d\lambda} T^{-1}(\lambda) \right|_{\lambda=\frac{\eta}{2}} + \text{const.} \quad (2.4)$$

Here

$$\mathcal{T}(\lambda) = \text{tr } T(\lambda) = A(\lambda) + D(\lambda). \quad (2.5)$$

Later on we shall consider the inhomogeneous XXZ model, for which

$$L_m(\lambda) = L_m(\lambda, \xi_m) = R_{0m}(\lambda - \xi_m), \quad T(\lambda) = L_M(\lambda, \xi_M) \dots L_2(\lambda, \xi_2) L_1(\lambda, \xi_1), \quad (2.6)$$

where ξ_m are arbitrary complex numbers attached to each lattice site that are called inhomogeneity parameters. In the homogeneous limit $\xi_m = \eta/2$ and we come back to the original model.

The commutation relations between the entries of the monodromy matrix are defined by the Yang-Baxter quadratic relations,

$$R_{12}(\lambda_1 - \lambda_2) T_1(\lambda_1) T_2(\lambda_2) = T_2(\lambda_2) T_1(\lambda_1) R_{12}(\lambda_1 - \lambda_2). \quad (2.7)$$

The equation (2.7) holds in the space $V_1 \otimes V_2 \otimes \mathcal{H}$ (where $V_j \sim \mathbb{C}^2$). The matrix $T_j(\lambda)$ acts in a nontrivial way in the space $V_j \otimes \mathcal{H}$, while the R -matrix is nontrivial in $V_1 \otimes V_2$.

In the framework of the algebraic Bethe ansatz an arbitrary quantum state can be obtained from the states generated by the action of the operators $B(\lambda)$ on the reference state $|0\rangle$ with all spins up,

$$|\psi\rangle = \prod_{j=1}^N B(\lambda_j) |0\rangle, \quad N = 0, 1, \dots, M. \quad (2.8)$$

The eigenstates of the transfer matrix $\mathcal{T}(\mu)$ can be constructed in the form (2.8), where the parameters λ_j satisfy the system of Bethe equations

$$a(\lambda_j) \prod_{\substack{k=1 \\ k \neq j}}^N \sinh(\lambda_k - \lambda_j + \eta) = d(\lambda_j) \prod_{\substack{k=1 \\ k \neq j}}^N \sinh(\lambda_k - \lambda_j - \eta), \quad j = 1, \dots, N, \quad (2.9)$$

and $a(\lambda)$, $d(\lambda)$ are the eigenvalues of the operators $A(\lambda)$ and $D(\lambda)$ on the reference state. In the normalization (2.1)–(2.2), we have

$$a(\lambda) = \prod_{a=1}^M \sinh(\lambda - \xi_a + \eta), \quad (2.10)$$

$$d(\lambda) = \prod_{a=1}^M \sinh(\lambda - \xi_a). \quad (2.11)$$

Remark. Generally speaking the system of equations (2.9) is neither a necessary nor a sufficient condition for the vector (2.8) to be an eigenstate of the transfer matrix. For example, the state (2.8) with all spins down ($N = M$) is the eigenstate of \mathcal{T} for generic $\{\lambda\}$. On the other hand the system (2.9) possesses solutions, which do not correspond to any eigenstate of \mathcal{T} . We discuss all these questions in more details in Section 3 and Appendix A.

The eigenvalue $\tau(\mu|\{\lambda\})$ of the operator $\mathcal{T}(\mu)$ corresponding to an eigenstate of the form (2.8) is

$$\tau(\mu|\{\lambda\}) = a(\mu) \prod_{k=1}^N \frac{\sinh(\lambda_k - \mu + \eta)}{\sinh(\lambda_k - \mu)} + d(\mu) \prod_{k=1}^N \frac{\sinh(\mu - \lambda_k + \eta)}{\sinh(\mu - \lambda_k)}, \quad (2.12)$$

The dual states can be constructed similarly to (2.8) via the operators $C(\lambda)$

$$\langle\psi| = \langle 0| \prod_{j=1}^N C(\lambda_j), \quad N = 0, 1, \dots, M. \quad (2.13)$$

Here $\langle 0| = |0\rangle^+$ and the dual eigenstates of $\mathcal{T}(\mu)$ are given in the form (2.13), where the parameters λ_j satisfy the same system of equations (2.9). Generically $\langle\psi| \neq |\psi\rangle^\dagger$ due to the involution $C(\lambda) = \pm B^\dagger(\bar{\lambda})$. If, however, the state (2.8) is the ground state of the Hamiltonian (1.3), then $\langle\psi| = c^N |\psi\rangle^\dagger$ with $c = 1$ for $\Delta > 1$ and $c = -1$ for $-1 < \Delta \leq 1$ respectively.

In the end of this section we give the explicit representations of the local spin operators in terms of the entries of the monodromy matrix. Such a representation is given by the solution of the quantum inverse scattering problem [25, 27]

$$\sigma_j^\alpha = \prod_{k=1}^{j-1} \mathcal{T}(\xi_k) \cdot \text{tr}(T(\xi_j) \sigma^\alpha) \cdot \prod_{k=1}^j \mathcal{T}^{-1}(\xi_k). \quad (2.14)$$

Here σ^α in the r.h.s. acts in the auxiliary space of $T(\lambda)$, while σ_j^α in the l.h.s. acts in the local quantum space \mathcal{H}_j . The formulas of the quantum inverse scattering problem permit us to embed the problem of calculation of the correlation functions of the local spin operators into the algebraic Bethe ansatz.

3 Twisted transfer matrix and scalar products

In this section we introduce new objects and notations, which will be used through all the paper. Starting from this section we consider only the subspace $\mathcal{H}^{(M/2-N)}$ of the quantum space \mathcal{H} with fixed (but arbitrary) number of spins down N .

First, we introduce the ‘twisted transfer matrix’ $\mathcal{T}_\kappa(\mu)$

$$\mathcal{T}_\kappa(\mu) = A(\mu) + \kappa D(\mu), \quad (3.1)$$

where κ is a complex parameter. We denote also as $|\psi_\kappa(\{\lambda\})\rangle$ and $\langle\psi_\kappa(\{\lambda\})|$ the eigenstates (respectively the dual eigenstates) of the operator $\mathcal{T}_\kappa(\mu)$ in the subspace $\mathcal{H}^{(M/2-N)}$. The corresponding eigenvalue is denoted as $\tau_\kappa(\mu|\{\lambda\})$. The same notations without a subscript κ correspond to the case $\kappa = 1$ considered in the previous section.

The eigenstates of $\mathcal{T}_\kappa(\mu)$ (and their dual states) have the form

$$|\psi_\kappa(\{\lambda\})\rangle = \prod_{j=1}^N B(\lambda_j)|0\rangle, \quad \langle\psi_\kappa(\{\lambda\})| = \langle 0| \prod_{j=1}^N C(\lambda_j), \quad (3.2)$$

with an eigenvalue

$$\tau_\kappa(\mu|\{\lambda\}) = a(\mu) \prod_{k=1}^N \frac{\sinh(\lambda_k - \mu + \eta)}{\sinh(\lambda_k - \mu)} + \kappa d(\mu) \prod_{k=1}^N \frac{\sinh(\mu - \lambda_k + \eta)}{\sinh(\mu - \lambda_k)}, \quad (3.3)$$

where the parameters $\{\lambda\}$ satisfy the system of twisted Bethe equations

$$a(\lambda_j) \prod_{\substack{k=1 \\ k \neq j}}^N \sinh(\lambda_k - \lambda_j + \eta) = \kappa d(\lambda_j) \prod_{\substack{k=1 \\ k \neq j}}^N \sinh(\lambda_k - \lambda_j - \eta), \quad j = 1, \dots, N. \quad (3.4)$$

It is also convenient to introduce the function

$$\begin{aligned} \mathcal{Y}_\kappa(\mu|\{\lambda\}) &= \prod_{k=1}^N \sinh(\lambda_k - \mu) \cdot \tau_\kappa(\mu|\{\lambda\}) \\ &= a(\mu) \prod_{k=1}^N \sinh(\lambda_k - \mu + \eta) + \kappa d(\mu) \prod_{k=1}^N \sinh(\lambda_k - \mu - \eta). \end{aligned} \quad (3.5)$$

In terms of this function the system (3.4) reads

$$\mathcal{Y}_\kappa(\lambda_j|\{\lambda\}) = 0, \quad j = 1, \dots, N. \quad (3.6)$$

Just like in the case $\kappa = 1$, for generic κ not all the solutions of the system (3.6) correspond to eigenvectors of $\mathcal{T}_\kappa(\mu)$. A brief sketch of the properties of the solutions of (3.6) is given in Appendix A (for details we refer the reader to the original paper [44]). Here we merely recall some definitions and one of the main results of [44].

Definition 3.1. A solution $\{\lambda\}$ of the system (3.6) is called *admissible*, if

$$d(\lambda_j) \prod_{\substack{k=1 \\ k \neq j}}^N \sinh(\lambda_j - \lambda_k + \eta) \neq 0, \quad j = 1, \dots, N, \quad (3.7)$$

and *unadmissible* otherwise. A solution is called *off-diagonal* if the parameters $\{\lambda\}$ are pair-wise distinct and *diagonal* otherwise. A solution is called *degenerated*, if the Jacobian of the system (3.6) vanishes, and *non-degenerated* otherwise. A solution is called *trivial*, if the corresponding state $|\psi_\kappa(\{\lambda\})\rangle$ is the zero vector.

One of the main result of [44] is that, for generic κ and $\{\xi\}$, the eigenstates of $\mathcal{T}_\kappa(\mu)$ corresponding to the admissible off-diagonal solutions of (3.6) form a basis in the space $\mathcal{H}^{(M/2-N)}$. Generically this basis is not normalized and non-orthogonal, although it is orthogonal to the dual basis. We would like to mention that for some specific choice of κ and $\{\xi\}$ certain inadmissible solutions may also contribute to the basis of the eigenstates in $\mathcal{H}^{(M/2-N)}$.

Consider now the scalar products of eigenstates (3.2) and arbitrary states of the form (2.8). The explicit results for such scalar products at $\kappa = 1$ were obtained in [45, 25]. Applying the methods used in these papers one can easily prove

Proposition 3.1. Let $\{\lambda\}$ satisfy the system (3.6), $\{\mu\}$ be generic complex numbers. Then

$$\begin{aligned} \langle 0 | \prod_{j=1}^N C(\mu_j) | \psi_\kappa(\{\lambda\}) \rangle &= \langle \psi_\kappa(\{\lambda\}) | \prod_{j=1}^N B(\mu_j) | 0 \rangle \\ &= \prod_{a=1}^N d(\lambda_a) \cdot \mathcal{X}_N^{-1}(\{\mu\}, \{\lambda\}) \cdot \det_N \left(\frac{\partial}{\partial \lambda_j} \tau_\kappa(\mu_k | \{\lambda\}) \right). \end{aligned} \quad (3.8)$$

Here $\mathcal{X}_N(\{\mu\}, \{\lambda\})$ is the Cauchy determinant composed of the parameters $\{\lambda\}$ and $\{\mu\}$

$$\mathcal{X}_N(\{\mu\}, \{\lambda\}) = \det_N \left(\frac{1}{\sinh(\mu_k - \lambda_j)} \right) = \frac{\prod_{a>b}^N \sinh(\lambda_a - \lambda_b) \sinh(\mu_b - \mu_a)}{\prod_{a,b=1}^N \sinh(\mu_b - \lambda_a)}. \quad (3.9)$$

To make these formulas more explicit we introduce for arbitrary positive integers n and n' ($n \leq n'$) and arbitrary sets of variables $\lambda_1, \dots, \lambda_n$, μ_1, \dots, μ_n and $\nu_1, \dots, \nu_{n'}$, such that $\{\lambda\} \subset \{\nu\}$, the following $n \times n$ matrix $\Omega_\kappa(\{\lambda\}, \{\mu\} | \{\nu\})$

$$\begin{aligned} (\Omega_\kappa)_{jk}(\{\lambda\}, \{\mu\} | \{\nu\}) &= a(\mu_k) t(\lambda_j, \mu_k) \prod_{a=1}^{n'} \sinh(\nu_a - \mu_k + \eta) \\ &\quad - \kappa d(\mu_k) t(\mu_k, \lambda_j) \prod_{a=1}^{n'} \sinh(\nu_a - \mu_k - \eta), \end{aligned} \quad (3.10)$$

with

$$t(\lambda, \mu) = \frac{\sinh \eta}{\sinh(\lambda - \mu) \sinh(\lambda - \mu + \eta)}. \quad (3.11)$$

Then the equation (3.8) reads

$$\begin{aligned} \langle 0 | \prod_{j=1}^N C(\mu_j) | \psi_\kappa(\{\lambda\}) \rangle &= \langle \psi_\kappa(\{\lambda\}) | \prod_{j=1}^N B(\mu_j) | 0 \rangle \\ &= \frac{\prod_{a=1}^N d(\lambda_a)}{\prod_{\substack{a,b=1 \\ a \neq b}}^N \sinh(\lambda_a - \lambda_b) \sinh(\mu_b - \mu_a)} \cdot \det_N \Omega_\kappa(\{\lambda\}, \{\mu\} | \{\lambda\}). \end{aligned} \quad (3.12)$$

Taking the limit $\{\mu\} \rightarrow \{\lambda\}$ in (3.12) we obtain the 'square of the norm' of the eigenstate (recall that generically $\langle \psi_\kappa(\{\lambda\}) | \neq | \psi_\kappa(\{\lambda\}) \rangle^\dagger$) [47, 48]

$$\langle \psi_\kappa(\{\lambda\}) | \psi_\kappa(\{\lambda\}) \rangle = \frac{\prod_{a=1}^N d(\lambda_a)}{\prod_{\substack{a,b=1 \\ a \neq b}}^N \sinh(\lambda_a - \lambda_b)} \cdot \det_N \Omega_\kappa(\{\lambda\}, \{\lambda\} | \{\lambda\}). \quad (3.13)$$

This equation can also be written in terms of a Jacobian using

$$\det_N \Omega_\kappa(\{\lambda\}, \{\lambda\} | \{\lambda\}) = \det_N \left(-\frac{\partial}{\partial \lambda_k} \mathcal{Y}_\kappa(\lambda_j | \{\lambda\}) \right), \quad (3.14)$$

where $\{\lambda\}$ satisfy the system (3.6).

4 Master equation for σ^z correlation function

In this paper we consider the generating functional for the correlation function of the third components of spins $\langle \sigma_1^z \sigma_{m+1}^z \rangle$. Following the papers [46, 39] we define for any complex number κ the operator $Q_{1,m}^\kappa$ as¹

$$Q_{1,m}^\kappa = \prod_{n=1}^m \left(\frac{1 + \kappa}{2} + \frac{1 - \kappa}{2} \cdot \sigma_n^z \right). \quad (4.1)$$

The generating functional is equal to the expectation value

$$\langle Q_{1,m}^\kappa \rangle = \frac{\langle \psi(\{\lambda\}) | Q_{1,m}^\kappa | \psi(\{\lambda\}) \rangle}{\langle \psi(\{\lambda\}) | \psi(\{\lambda\}) \rangle}, \quad (4.2)$$

where $|\psi(\{\lambda\})\rangle$ is an eigenstate of $\mathcal{T}(\mu)$ in the subspace $\mathcal{H}^{(M/2-N)}$. Taking the second 'lattice derivative' of $\langle Q_{1,m}^\kappa \rangle$ and the second derivative with respect to κ at $\kappa = 1$ we extract the

¹Setting $\kappa = e^\beta$ one has $Q_{1,m}^\kappa = \exp(\frac{\beta}{2} \sum_{n=1}^m (1 - \sigma_n^z))$ and, hence, this is exactly the operator considered in [31].

two-point correlation function of the third components of local spins:

$$\frac{1}{2} \langle (1 - \sigma_1^z)(1 - \sigma_{m+1}^z) \rangle = \frac{\partial^2}{\partial \kappa^2} \langle (Q_{1,m+1}^\kappa - Q_{1,m}^\kappa - Q_{2,m+1}^\kappa + Q_{2,m}^\kappa) \rangle \Big|_{\kappa=1}. \quad (4.3)$$

The main result of this paper is an integral representation for the generating functional (4.2).

Theorem 4.1. *Let the inhomogeneities $\{\xi\}$ be generic and the set $\{\lambda\}$ be an admissible off-diagonal solution of the system (2.9). Then there exists $\kappa_0 > 0$ such, that for $|\kappa| < \kappa_0$ the expectation value of the operator $Q_{1,m}^\kappa$ in the inhomogeneous finite XXZ chain is given by the multiple contour integral*

$$\begin{aligned} \langle Q_{1,m}^\kappa \rangle = & \frac{1}{N!} \oint_{\Gamma\{\xi\} \cup \Gamma\{\lambda\}} \prod_{j=1}^N \frac{dz_j}{2\pi i} \cdot \prod_{a,b=1}^N \sinh^2(\lambda_a - z_b) \cdot \prod_{a=1}^m \frac{\tau_\kappa(\xi_a|\{z\})}{\tau(\xi_a|\{\lambda\})} \\ & \times \frac{\det_N \left(\frac{\partial \tau_\kappa(\lambda_j|\{z\})}{\partial z_k} \right) \cdot \det_N \left(\frac{\partial \tau(z_k|\{\lambda\})}{\partial \lambda_j} \right)}{\prod_{a=1}^N \mathcal{Y}_\kappa(z_a|\{z\}) \cdot \det_N \left(\frac{\partial \mathcal{Y}(\lambda_k|\{\lambda\})}{\partial \lambda_j} \right)}. \end{aligned} \quad (4.4)$$

The integration contour is such that the only singularities of the integrand (4.4) within $\Gamma\{\xi\} \cup \Gamma\{\lambda\}$ which contribute to the integral are the points $\{\xi\}$ and $\{\lambda\}$.

We call (4.4) the master equation.

Remark 1. The master equation (4.4) gives the expectation value $\langle Q_{1,m}^\kappa \rangle$ with respect to an arbitrary eigenstate $|\psi(\{\lambda\})\rangle$ of \mathcal{T} . In particular one can chose $\{\lambda\}$ such that in the homogeneous limit $\xi_a \rightarrow \eta/2$ the corresponding eigenstate $|\psi(\{\lambda\})\rangle$ goes to the ground state of the XXZ Hamiltonian.

Remark 2. The master equation (4.4) is an integral representation of the expectation value $\langle Q_{1,m}^\kappa \rangle$, which holds true at least for $|\kappa|$ small enough. On the other hand this expectation value is evidently a polynomial in κ of degree m . Therefore the representation (4.4) can be easily continued in κ from any vicinity of the origin to the whole complex plane. This does not mean, however, that one can set κ to be an arbitrary specific value directly in the integrand of (4.4).

Proof of Theorem 4.1. In order to compute the expectation value of $Q_{1,m}^\kappa$, we express this operator in terms of the twisted transfer matrix. Due to (2.14) one has

$$Q_{1,m}^\kappa = \prod_{a=1}^m \mathcal{T}_\kappa(\xi_a) \prod_{b=1}^m \mathcal{T}^{-1}(\xi_b). \quad (4.5)$$

Then we can use the explicit formula for the multiple action of $\prod_{a=1}^m \mathcal{T}_\kappa(x_a)$ for an arbitrary set

of complex numbers $\{x\}$ on an arbitrary state $\langle 0 | \prod_{j=1}^N C(\lambda_j)$, [31]

$$\begin{aligned} & \langle 0 | \prod_{j=1}^N C(\lambda_j) \prod_{a=1}^m \mathcal{T}_\kappa(x_a) \\ &= \sum_{n=0}^{\min(m, N)} \sum_{\substack{\{\lambda\} = \{\lambda_{\alpha_+}\} \cup \{\lambda_{\alpha_-}\} \\ \{x\} = \{x_{\gamma_+}\} \cup \{x_{\gamma_-}\} \\ |\alpha_+| = |\gamma_+| = n}} R_n^\kappa(\{x_{\gamma_+}\}, \{x_{\gamma_-}\}, \{\lambda_{\alpha_+}\}, \{\lambda_{\alpha_-}\}) \langle 0 | \prod_{a \in \gamma_+} C(x_a) \prod_{b \in \alpha_-} C(\lambda_b), \end{aligned} \quad (4.6)$$

where the coefficient $R_n^\kappa(\{x_{\gamma_+}\}, \{x_{\gamma_-}\}, \{\lambda_{\alpha_+}\}, \{\lambda_{\alpha_-}\})$ is given by

$$\begin{aligned} R_n^\kappa = & \left\{ \prod_{\substack{a > b \\ a, b \in \alpha_+}} \sinh(\lambda_b - \lambda_a) \prod_{\substack{a < b \\ a, b \in \gamma_+}} \sinh(x_b - x_a) \prod_{a \in \alpha_+} \prod_{b \in \alpha_-} \sinh(\lambda_b - \lambda_a) \right\}^{-1} \\ & \times \prod_{a \in \gamma_-} \tau_\kappa(x_a | \{x_{\gamma_+}\} \cup \{\lambda_{\alpha_-}\}) \cdot \det_n \Omega_\kappa(\{x_{\gamma_+}\}, \{\lambda_{\alpha_+}\} | \{x_{\gamma_+}\} \cup \{\lambda_{\alpha_-}\}). \end{aligned} \quad (4.7)$$

Here $\Omega_\kappa(\{x_{\gamma_+}\}, \{\lambda_{\alpha_+}\} | \{x_{\gamma_+}\} \cup \{\lambda_{\alpha_-}\})$ is given by expressions (3.10) in which the sets $\{\lambda\}$, $\{\mu\}$, $\{\nu\}$ have to be replaced by $\{x_{\gamma_+}\}$, $\{\lambda_{\alpha_+}\}$ and $\{x_{\gamma_+}\} \cup \{\lambda_{\alpha_-}\}$ respectively.

The summation in (4.6) is taken with respect to partitions of the sets $\{\lambda\}$ and $\{x\}$ into disjoint subsets $\{\lambda\} = \{\lambda_{\alpha_+}\} \cup \{\lambda_{\alpha_-}\}$ and $\{x\} = \{x_{\gamma_+}\} \cup \{x_{\gamma_-}\}$, such that the number of elements in the subsets $\{\lambda_{\alpha_+}\}$ and $\{x_{\gamma_+}\}$ coincides and is equal to n .

In the paper [31] we directly applied the equation (4.6) for the computation of $\langle Q_{1,m}^\kappa \rangle$, specifying $x_j = \xi_j$ for $j = 1, \dots, m$. The peculiarity of the inhomogeneity parameters is that $d(\xi_j) = 0$, what simplifies the formulas. In the present paper, however, we are going to keep the parameters $\{x\}$ arbitrary, i.e. we shall consider the expectation value of the operator

$$\mathcal{Q}_m(\kappa, \{x\}) = \prod_{a=1}^m \mathcal{T}_\kappa(x_a) \cdot \prod_{b=1}^m \mathcal{T}^{-1}(x_b), \quad (4.8)$$

where the parameters x_1, \dots, x_m are generic complex numbers.

The normalized expectation value of $\mathcal{Q}_m(\kappa, \{x\})$ on the finite lattice with respect to an arbitrary eigenstate of the transfer matrix $|\psi(\{\lambda\})\rangle$ has the form

$$\langle \mathcal{Q}_m(\kappa, \{x\}) \rangle = \frac{\langle \psi(\{\lambda\}) | \prod_{a=1}^m \mathcal{T}_\kappa(x_a) \cdot \prod_{b=1}^m \mathcal{T}^{-1}(x_b) | \psi(\{\lambda\}) \rangle}{\langle \psi(\{\lambda\}) | \psi(\{\lambda\}) \rangle}. \quad (4.9)$$

The action to the right of the product of $\mathcal{T}^{-1}(x_b)$ produces merely a numerical factor:

$$\prod_{b=1}^m \mathcal{T}^{-1}(x_b) | \psi(\{\lambda\}) \rangle = \prod_{b=1}^m \tau^{-1}(x_b | \{\lambda\}) \cdot | \psi(\{\lambda\}) \rangle. \quad (4.10)$$

Acting to the left with the product of $\mathcal{T}_\kappa(x_a)$ by means of (4.6) with $m < N$, and using the expression (3.12) of the scalar product at $\kappa = 1$, we obtain

$$\langle \mathcal{Q}_m(\kappa, \{x\}) \rangle = \sum_{n=0}^m \sum_{\substack{\{\lambda\}=\{\lambda_{\alpha_+}\} \cup \{\lambda_{\alpha_-}\} \\ \{x\}=\{x_{\gamma_+}\} \cup \{x_{\gamma_-}\} \\ |\alpha_+|=|\gamma_+|=n}} \Delta_n^\kappa(\{x_{\gamma_+}\}, \{x_{\gamma_-}\}, \{\lambda_{\alpha_+}\}, \{\lambda_{\alpha_-}\}), \quad (4.11)$$

with

$$\begin{aligned} \Delta_n^\kappa &= \prod_{b=1}^m \left\{ \tau(x_b | \{\lambda\}) \prod_{a \in \alpha_-} \sinh(\lambda_a - x_b) \prod_{\substack{a \in \gamma_+ \\ a \neq b}} \sinh(x_a - x_b) \right\}^{-1} \prod_{a \in \gamma_-} \mathcal{Y}_\kappa(x_a | \{x_{\gamma_+}\} \cup \{\lambda_{\alpha_-}\}) \\ &\times \det_n \Omega_\kappa(\{x_{\gamma_+}\}, \{\lambda_{\alpha_+}\} | \{x_{\gamma_+}\} \cup \{\lambda_{\alpha_-}\}) \cdot \frac{\det_N \Omega(\{\lambda\}, \{x_{\gamma_+}\} \cup \{\lambda_{\alpha_-}\} | \{\lambda\})}{\det_N \Omega(\{\lambda\}, \{\lambda_{\alpha_+}\} \cup \{\lambda_{\alpha_-}\} | \{\lambda\})}. \end{aligned} \quad (4.12)$$

Here \mathcal{Y} and Ω without subscript κ are equal respectively to \mathcal{Y}_κ and Ω_κ at $\kappa = 1$. The elements in the sets $\{x_{\gamma_+}\} \cup \{\lambda_{\alpha_-}\}$ and $\{\lambda_{\alpha_+}\} \cup \{\lambda_{\alpha_-}\}$ are ordered accordingly.

Like in [31], we can now perform a re-summation over the partitions of the set $\{x\}$ by introducing contour integrals over auxiliary variables z_1, \dots, z_n :

$$\begin{aligned} \langle \mathcal{Q}_m(\kappa, \{x\}) \rangle &= \sum_{n=0}^m \sum_{\substack{\{\lambda\}=\{\lambda_{\alpha_+}\} \cup \{\lambda_{\alpha_-}\} \\ |\alpha_+|=n}} \prod_{a=1}^m \prod_{b \in \alpha_-} \sinh^{-1}(\lambda_b - x_a) \cdot \frac{1}{n!} \oint_{\Gamma\{x\}} \prod_{j=1}^n \frac{dz_j}{2\pi i} \\ &\times \prod_{a=1}^n \prod_{b=1}^m \frac{1}{\sinh(z_a - x_b)} \cdot \prod_{a=1}^m \frac{\mathcal{Y}_\kappa(x_a | \{z\} \cup \{\lambda_{\alpha_-}\})}{\tau(x_a | \{\lambda\})} \cdot \prod_{a=1}^n \frac{1}{\mathcal{Y}_\kappa(z_a | \{z\} \cup \{\lambda_{\alpha_-}\})} \\ &\times \det_n \Omega_\kappa(\{z\}, \{\lambda_{\alpha_+}\} | \{z\} \cup \{\lambda_{\alpha_-}\}) \cdot \frac{\det_N \Omega(\{\lambda\}, \{z\} \cup \{\lambda_{\alpha_-}\} | \{\lambda\})}{\det_N \Omega(\{\lambda\}, \{\lambda_{\alpha_+}\} \cup \{\lambda_{\alpha_-}\} | \{\lambda\})}, \end{aligned} \quad (4.13)$$

where the closed contour $\Gamma\{x\}$ surrounds the points¹ x_1, \dots, x_m and do not contain any other pole of the integrand, i.e. zeros in \mathbb{C}^n of the functions $\mathcal{Y}_\kappa(z_a | \{z\} \cup \{\lambda_{\alpha_-}\})$. Since at this stage of the computations x_1, \dots, x_m are generic complex numbers, we can always choose them separated from the zeros of $\mathcal{Y}_\kappa(z_a | \{z\} \cup \{\lambda_{\alpha_-}\})$ for any subset $\{\lambda_{\alpha_-}\}$.

Thus, the sum over partitions of the set $\{x\}$ in (4.11) is replaced with a contour integral. In paper [31] we replaced in the thermodynamical limit the sum over partitions of the set $\{\lambda\}$ with the integrals over the support of the spectral density of the ground state. Here we treat this sum in a different way. Namely, we perform a second re-summation over the partitions of

¹More precisely, $\Gamma\{x\}$ is the boundary of a set of polydisks $\mathcal{D}_a(r)$ in \mathbb{C}^n . Namely, $\Gamma\{x\} = \cup_{a=1}^m \bar{\mathcal{D}}_a(r)$, where $\bar{\mathcal{D}}_a(r) = \{z \in \mathbb{C}^n : |z_k - x_a| = r, \quad k = 1, \dots, n\}$. The integration contour $\Gamma\{\xi\} \cup \Gamma\{\lambda\}$ in (4.4) should be understood in a similar manner in \mathbb{C}^N .

the set $\{\lambda\}$ in (4.13) also in terms of a contour integral. Indeed, one can easily see that

$$\begin{aligned} & \text{Res}_{\{z_{n+1}, \dots, z_N\}=\{\lambda_{\alpha_-}\}} \left[\det_N \Omega_\kappa(\{z_1, \dots, z_N\}, \{\lambda_{\alpha_+}\} \cup \{\lambda_{\alpha_-}\} | \{z_1, \dots, z_N\}) \right] \\ &= \prod_{a \in \alpha_-} \mathcal{Y}_\kappa(\lambda_a | \{z_1, \dots, z_n\} \cup \{\lambda_{\alpha_-}\}) \cdot \det_n \Omega_\kappa(\{z_1, \dots, z_n\}, \{\lambda_{\alpha_+}\} | \{z_1, \dots, z_n\} \cup \{\lambda_{\alpha_-}\}). \end{aligned} \quad (4.14)$$

This enables us to express the generating functional $\langle \mathcal{Q}_m(\kappa, \{x\}) \rangle$ for the finite XXZ chain as a single multiple integral:

$$\begin{aligned} \langle \mathcal{Q}_m(\kappa, \{x\}) \rangle &= \frac{1}{N!} \oint_{\Gamma\{x\} \cup \Gamma\{\lambda\}} \prod_{j=1}^N \frac{dz_j}{2\pi i} \cdot \prod_{a=1}^m \frac{\tau_\kappa(x_a | \{z\})}{\tau(x_a | \{\lambda\})} \cdot \prod_{a=1}^N \frac{1}{\mathcal{Y}_\kappa(z_a | \{z\})} \\ &\quad \times \det_N \Omega_\kappa(\{z\}, \{\lambda\} | \{z\}) \cdot \frac{\det_N \Omega(\{\lambda\}, \{z\} | \{\lambda\})}{\det_N \Omega(\{\lambda\}, \{\lambda\} | \{\lambda\})}, \end{aligned} \quad (4.15)$$

where the closed contour $\Gamma\{x\} \cup \Gamma\{\lambda\}$ surrounds the points x_1, \dots, x_m and $\lambda_1, \dots, \lambda_N$ (spectral parameters corresponding to given eigenstate of $\mathcal{T}(\mu)$) and no other poles of the integrand.

Finally, in order to reproduce the expectation value $\langle Q_{1,m}^\kappa \rangle$ we should set the generic parameters $\{x\}$ in (4.15) to be equal to the inhomogeneities ξ_1, \dots, ξ_m . This definitely can be done if the system

$$\mathcal{Y}_\kappa(z_j | \{z\}) = 0, \quad j = 1, \dots, N \quad (4.16)$$

has no solution inside the integration contour $\Gamma\{\xi\} \cup \Gamma\{\lambda\}$. Thus, we need to analyze the properties of the solutions of the system of the twisted Bethe equations. Hereby, since $\langle \mathcal{Q}_m(\kappa, \{x\}) \rangle$ is a polynomial in κ , it is sufficient to determine this polynomial in an open ball around some specific value, for example for $|\kappa|$ small enough.

The detail analysis of the system (4.16) was done in the paper [44]. We give some of the basic statements of this paper in Appendix A. Here we present only the results necessary for writing and evaluating the integral (4.15).

Lemma 4.1. *Let the inhomogeneities ξ_1, \dots, ξ_m be generic, $\lambda_1, \dots, \lambda_N$ be an admissible off-diagonal solution of (2.9) and $|\kappa|$ be small enough. Then all admissible off-diagonal solutions of the system (4.16) are separated from the points $\{\xi\}$ and $\{\lambda\}$.*

Corollary 4.1. *There exists a contour $\Gamma\{\xi\} \cup \Gamma\{\lambda\}$ such that the points $\{\xi\}$ and $\{\lambda\}$ are inside this contour, while all admissible off-diagonal solutions of the system (4.16) are outside $\Gamma\{\xi\} \cup \Gamma\{\lambda\}$.*

Lemma 4.2. *Let the contour $\Gamma\{\xi\} \cup \Gamma\{\lambda\}$ satisfy the conditions of the corollary 4.1, and $x_k \rightarrow \xi_k$. Then:*

- 1) *the only poles inside the contour $\Gamma\{\xi\} \cup \Gamma\{\lambda\}$ which provide non-vanishing contribution to the integral (4.15) are in $\{\xi\} \cup \{\lambda\}$;*
- 2) *the only poles outside the contour $\Gamma\{\xi\} \cup \Gamma\{\lambda\}$ which provide non-vanishing contribution to the integral (4.15) are the admissible off-diagonal solutions of the system (4.16).*

The proof of these lemmas is given in the Appendix A using the results of [44].

Lemma 4.2 guarantees that for $x_k = \xi_k$ there are no any other poles of the integrand (4.15) inside the contour $\Gamma\{\xi\} \cup \Gamma\{\lambda\}$ and contributing to the integral except $z_j = \xi_k$ and $z_j = \lambda_k$. Thus, setting $x_k = \xi_k$ and using $d(\xi_k) = 0$ we arrive at

$$\begin{aligned} \langle Q_{1,m}^\kappa \rangle = \frac{1}{N!} \oint_{\Gamma\{\xi\} \cup \Gamma\{\lambda\}} \prod_{j=1}^N \frac{dz_j}{2\pi i} \cdot \prod_{a=1}^N \prod_{b=1}^m \left(\frac{\sinh(\lambda_a - \xi_b)}{\sinh(z_a - \xi_b)} \cdot \frac{\sinh(z_a - \xi_b + \eta)}{\sinh(\lambda_a - \xi_b + \eta)} \right) \\ \times \frac{\det_N \Omega_\kappa(\{z\}, \{\lambda\}|\{z\}) \cdot \det_N \Omega(\{\lambda\}, \{z\}|\{\lambda\})}{\prod_{a=1}^N \mathcal{Y}_\kappa(z_a|\{z\}) \cdot \det_N \Omega(\{\lambda\}, \{\lambda\}|\{\lambda\})}, \quad (4.17) \end{aligned}$$

It remains to express the determinants of the matrices Ω_κ and Ω in terms of Jacobians using (3.8)–(3.14), and we obtain the master equation (4.4). \square

5 Multiple integral representation

Lemma 4.2 permits us to evaluate the integral (4.4) in two different ways: either to compute the residues of the integrand inside the contour $\Gamma\{\xi\} \cup \Gamma\{\lambda\}$, or to compute the residues outside this contour. In this section we consider the first way, which immediately leads us to the representation for $\langle Q_{1,m}^\kappa \rangle$ obtained in [31].

For this purpose it is more convenient to use (4.17), which in fact is equivalent to (4.4). This multiple integral can be presented as

$$\oint_{\Gamma\{\xi\} \cup \Gamma\{\lambda\}} \prod_{j=1}^N dz_j = \sum_{n=0}^N C_N^n \oint_{\Gamma\{\xi\}} \prod_{j=1}^n dz_j \oint_{\Gamma\{\lambda\}} \prod_{j=1}^{N-n} dz_j. \quad (5.1)$$

Hereby, since the number of the poles inside $\Gamma\{\xi\}$ is m and the integrand vanishes as soon as $z_j = z_k$, the sum in (5.1) is actually restricted to $n \leq m$. Evaluating $N - n$ integrals in the points $\{\lambda_{\alpha_-}\}$ we arrive at (4.13) with $x_k = \xi_k$, and hence $d(x_k) = 0$. Moreover, we can also set $d(z_k) = 0$, since the remaining integrals surround only the poles at $z_k = \xi_j$ and after evaluation of these integrals the functions $d(z_k)$ vanish. Finally, using the fact that the set $\{\lambda\}$ satisfies the system (2.9) we obtain

$$\begin{aligned} \langle Q_{1,m}^\kappa \rangle = \sum_{n=0}^m \sum_{\substack{\{\lambda\} = \{\lambda_{\alpha_+}\} \cup \{\lambda_{\alpha_-}\} \\ |\alpha_+| = n}} \frac{1}{n!} \oint_{\Gamma\{\xi\}} \prod_{j=1}^n \frac{dz_j}{2\pi i} \cdot \frac{\prod_{b \in \alpha_+} U(\lambda_b|\{z\}, \{\lambda_{\alpha_-}\}, \{\xi\})}{\prod_{b=1}^n U(z_b|\{z\}, \{\lambda_{\alpha_-}\}, \{\xi\})} \\ \times \det_n \tilde{M}_\kappa(\{\lambda_{\alpha_+}\}, \{z\}) \frac{\det_N \Omega(\{\lambda\}, \{z\} \cup \{\lambda_{\alpha_-}\}|\{\lambda\})|_{d(z)=0}}{\det_N \Omega(\{\lambda\}, \{\lambda_{\alpha_+}\} \cup \{\lambda_{\alpha_-}\}|\{\lambda\})}, \quad (5.2) \end{aligned}$$

where

$$U(\nu|\{z\}, \{\lambda_{\alpha_-}\}, \{\xi\}) = a(\nu) \prod_{a=1}^n \sinh(z_a - \nu + \eta) \prod_{a \in \alpha_-} \sinh(\lambda_a - \nu + \eta) \prod_{a=1}^m \frac{\sinh(\nu - \xi_a)}{\sinh(\nu - \xi_a + \eta)}, \quad (5.3)$$

and

$$(\tilde{M}_\kappa)_{jk}(\{\lambda\}, \{z\}) = t(z_k, \lambda_j) + \kappa t(\lambda_j, z_k) \prod_{a=1}^n \frac{\sinh(\lambda_a - \lambda_j + \eta)}{\sinh(\lambda_j - \lambda_a + \eta)} \frac{\sinh(\lambda_j - z_a + \eta)}{\sinh(z_a - \lambda_j + \eta)}. \quad (5.4)$$

Observe that, unlike in the master equation, the integrand in the r.h.s. of (5.2) is a polynomial in κ , since κ enters only the determinant of the matrix \tilde{M}_κ (5.4). Therefore one can set κ to be an arbitrary complex number directly in the integral representation (5.2).

This representation also allows one to proceed, if necessary, to the homogeneous limit simply by setting $\xi_k \rightarrow \eta/2$. Finally, the equation (5.2) is convenient for taking the thermodynamic limit $N, M \rightarrow \infty$ $N/M = \text{const}$. We refer the reader for the details to the paper [31] and here simply recall the final result for the ground state expectation value of $\langle Q_{1,m}^\kappa \rangle$ in the thermodynamic limit and in an external magnetic field in the massive and massless regimes:

$$\begin{aligned} \langle Q_{1,m}^\kappa \rangle &= \sum_{n=0}^m \frac{1}{(n!)^2} \oint_{\Gamma\{\xi\}} \prod_{j=1}^n \frac{dz_j}{2\pi i} \int_C d^n \lambda \prod_{b=1}^n \prod_{a=1}^m \left[\frac{\sinh(\lambda_b - \xi_a)}{\sinh(z_b - \xi_a)} \frac{\sinh(z_b - \xi_a + \eta)}{\sinh(\lambda_b - \xi_a + \eta)} \right] \\ &\times \prod_{a=1}^n \prod_{b=1}^n \frac{\sinh(\lambda_a - z_b + \eta)}{\sinh(\lambda_a - \lambda_b + \eta)} \frac{\sinh(z_b - \lambda_a + \eta)}{\sinh(z_a - z_b + \eta)} \cdot \det_n [\tilde{M}_\kappa(\{\lambda\}|\{z\})] \cdot \det_n [\rho(\lambda_j, z_k)]. \end{aligned} \quad (5.5)$$

Here the function $\rho(\lambda, z)$ is the ‘inhomogeneous spectral density’ of the ground state, having the support on the contour C as defined in [31].

6 Form factor expansion

Evaluating the integral (4.4) by the residues outside the integration contour we arrive at the expansion over form factors for $\langle Q_{1,m}^\kappa \rangle$. Recall that due to Lemma 4.2, the only poles outside the contour $\Gamma\{\xi\} \cup \Gamma\{\lambda\}$ which contribute to the integral (4.4), are the admissible off-diagonal solutions of (4.16). Hence

$$\begin{aligned} \langle Q_{1,m}^\kappa \rangle &= (-1)^N \sum_{\{\mu\}} \prod_{a,b=1}^N \sinh^2(\lambda_a - \mu_b) \cdot \prod_{a=1}^m \frac{\tau_\kappa(\xi_a|\{\mu\})}{\tau(\xi_a|\{\lambda\})} \\ &\times \frac{\det_N \left(\frac{\partial \tau_\kappa(\lambda_j|\{\mu\})}{\partial \mu_k} \right)}{\det_N \left(\frac{\partial \mathcal{Y}_\kappa(\mu_k|\{\mu\})}{\partial \mu_j} \right)} \cdot \frac{\det_N \left(\frac{\partial \tau(\mu_k|\{\lambda\})}{\partial \lambda_j} \right)}{\det_N \left(\frac{\partial \mathcal{Y}(\lambda_k|\{\lambda\})}{\partial \lambda_j} \right)}, \end{aligned} \quad (6.1)$$

where the sum is taken with respect to all admissible off-diagonal solutions μ_1, \dots, μ_N of the system (4.16). Due to the Theorem A.1 the Jacobian matrix $\partial \mathcal{Y}_\kappa(\mu_k|\{\mu\})/\partial \mu_j$ is non-degenerated.

Using the formula (3.8) we identify each Jacobian in (6.1) with the corresponding scalar products, leading to

$$\langle Q_{1,m}^\kappa \rangle = \sum_{\{\mu\}} \prod_{a=1}^m \frac{\tau_\kappa(\xi_a|\{\mu\})}{\tau(\xi_a|\{\lambda\})} \cdot \frac{\langle \psi(\{\lambda\})|\psi_\kappa(\{\mu\}) \rangle}{\langle \psi_\kappa(\{\mu\})|\psi_\kappa(\{\mu\}) \rangle} \cdot \frac{\langle \psi_\kappa(\{\mu\})|\psi(\{\lambda\}) \rangle}{\langle \psi(\{\lambda\})|\psi(\{\lambda\}) \rangle}. \quad (6.2)$$

It remains to use that the state $|\psi_\kappa(\{\mu\})\rangle$ is the eigenstate of $\mathcal{T}_\kappa(\xi)$ with the eigenvalue $\tau_\kappa(\xi|\{\mu\})$ and the state $|\psi(\{\lambda\})\rangle$ is the eigenstate of $\mathcal{T}(\xi)$ with the eigenvalue $\tau(\xi|\{\lambda\})$. Thus, the equation (6.2) can be written in the form

$$\langle Q_{1,m}^\kappa \rangle = \sum_{\{\mu\}} \frac{\langle \psi(\{\lambda\})|\prod_{b=1}^m \mathcal{T}_\kappa(\xi_b)|\psi_\kappa(\{\mu\}) \rangle \cdot \langle \psi_\kappa(\{\mu\})|\prod_{b=1}^m \mathcal{T}^{-1}(\xi_b)|\psi(\{\lambda\}) \rangle}{\langle \psi_\kappa(\{\mu\})|\psi_\kappa(\{\mu\}) \rangle \cdot \langle \psi(\{\lambda\})|\psi(\{\lambda\}) \rangle}. \quad (6.3)$$

Observe that we did not use the statement b) of the Theorem A.1 on the completeness of the set $|\psi_\kappa(\{\mu\})\rangle$. The sum over eigenstates of \mathcal{T}_κ appears automatically as the result of the evaluation of the multiple integral (4.4) by the residues outside the integration contour.

Taking the second lattice derivative of (6.1) and then differentiating twice with respect to κ at $\kappa = 1$ (see Appendix B) we obtain the form factor type expansion directly for the correlation function of the third components of the spin

$$\langle \sigma_1^z \sigma_{m+1}^z \rangle = \langle \sigma_1^z \rangle \cdot \langle \sigma_{m+1}^z \rangle + \sum_{\{\mu\} \neq \{\lambda\}} \frac{\langle \psi(\{\lambda\})|\sigma_1^z|\psi(\{\mu\}) \rangle \cdot \langle \psi(\{\mu\})|\sigma_{m+1}^z|\psi(\{\lambda\}) \rangle}{\langle \psi(\{\mu\})|\psi(\{\mu\}) \rangle \cdot \langle \psi(\{\lambda\})|\psi(\{\lambda\}) \rangle}, \quad (6.4)$$

where the form factors of σ^z are given by (B.10). Observe that here the summation is taken with respect to the eigenstates of the operator \mathcal{T} , but not \mathcal{T}_κ . In other words, in the homogeneous limit, this is the sum over the excited states of the Hamiltonian.

Of course, the equation (6.4) might be obtained by means of inserting the complete set of the eigenstates of the Hamiltonian between local spin operators. Such a representation, however, would be quite formal without a precise description of the set of the states on which the summation has to be performed. On the one hand it is clear that the sum should be taken with respect to solutions of the system (2.9), but on the other hand it is also clear that not any solution of this system corresponds to an eigenstate of the Hamiltonian. Note that one can not simply restrict the sum in (6.4) to the admissible off-diagonal solutions, since in the homogeneous limit certain inadmissible solutions of the system (2.9) give rise to the basis of the eigenstates.

The master equation approach gives the way to overcome these difficulties. Namely, if we use the approach (1.2), we can insert the set of the eigenstates of the twisted transfer matrix between local operators. Then the summation in (6.4) is restricted to the admissible off-diagonal solutions of the system (3.6). The inadmissible solutions corresponding to the eigenstates of the Hamiltonian appear in this case only as the limit of admissible solutions at $\kappa \rightarrow 1$. Since for $|\kappa|$ small enough all admissible solutions are in the vicinities of the points $\{\xi - \eta\}$, the sum over

excited states can be written as a multiple integral with respect to a contour surrounding these points. This allows one to obtain the master type equation directly for the spin–spin correlation functions, starting from their form factor expansions.

7 Conclusion

The main result of this paper is the master equation (4.4). We have shown that this equation draws a link between multiple integral representations and form factor expansions for the correlation functions.

The master equation sheds a new light on the role of the auxiliary contour integrals in the representations for the correlation functions derived in [31]. Originally they appeared as a result of a re-summation of the elementary blocks obtained in [22, 23, 26]. In the framework of the master equation approach the same auxiliary contour integrals are equivalent to the sum over excited states.

Using the master equation method one can derive multiple integral representations for other correlation functions of the XXZ model, including many-point correlators. Indeed, an arbitrary correlation function in the finite chain can be reduced to the multiple sum over complete set of the eigenstates of the twisted transfer matrix. On the other hand the explicit formulas for the form factors of the local spin operators obtained in [25] can be easily generalized for the case when one or both states are eigenstates of the twisted transfer matrix. Presenting this sum as a contour integral of the type (4.4) and evaluating this integral by the residues inside this contour we can express an arbitrary correlation function of the XXZ model as a multiple integral of the form (5.5).

Moreover, we conjecture that representations similar to (4.4) should exist for the correlation functions of other models solvable by the algebraic Bethe ansatz. This conjecture is based on the fact that the integrand of (4.4) depends mostly on the eigenvalues of the twisted transfer matrix, which is a typical object for the algebraic Bethe ansatz. As we have mentioned already in Introduction, it would be very desirable to obtain an analog of this master equation in the case of the field theory models. For these models such an integral representation could give an analytic link between short distance and long distance expansions of the correlation functions.

Finally, the master equation method opens a way to obtain multiple integral representations for time-dependent correlation functions. This subject will be considered in our forthcoming publication [49]. Here we only announce a generalization of the representation (5.5) for the time-dependent case in the massive and massless regimes

$$\langle \sigma_1^z(0) \sigma_{m+1}^z(t) \rangle = 2 \langle \sigma_1^z(0) \rangle - 1 + 2D_m^2 \frac{\partial^2}{\partial \kappa^2} \mathcal{Q}_\kappa(m, t) \Big|_{\kappa=1}, \quad (7.1)$$

where D_m^2 means the second lattice derivative and

$$\begin{aligned} \mathcal{Q}_\kappa(m, t) = & \sum_{n=0}^{\infty} \frac{1}{(n!)^2} \int_C d^n \lambda \oint_{\Gamma\{\pm \frac{\eta}{2}\}} \prod_{j=1}^n \frac{dz_j}{2\pi i} \cdot \prod_{a,b=1}^n \frac{\sinh(\lambda_a - z_b + \eta) \sinh(z_b - \lambda_a + \eta)}{\sinh(\lambda_a - \lambda_b + \eta) \sinh(z_a - z_b + \eta)} \\ & \times \prod_{b=1}^n e^{it(E(z_b) - E(\lambda_b)) + im(p(z_b) - p(\lambda_b))} \det_n \tilde{M}_\kappa(\{\lambda\}|\{z\}) \cdot \det_n [\mathcal{R}_n^\kappa(\lambda_j, z_k|\{\lambda\}, \{z\})]. \end{aligned} \quad (7.2)$$

Here \tilde{M}_κ is given by (5.4). The functions $E(\lambda)$ and $p(\lambda)$ are the bare one-particle energy and momentum

$$E(\lambda) = \frac{2 \sinh^2 \eta}{\sinh(\lambda + \frac{\eta}{2}) \sinh(\lambda - \frac{\eta}{2})}, \quad p(\lambda) = i \log \left(\frac{\sinh(\lambda - \frac{\eta}{2})}{\sinh(\lambda + \frac{\eta}{2})} \right). \quad (7.3)$$

The contour $\Gamma\{\pm \eta/2\}$ surrounds the points $\pm \eta/2$ and does not contain any other singularities of the integrand. The function $\mathcal{R}_n^\kappa(\lambda, z|\{\lambda\}, \{z\})$, as a function of z and other arguments fixed, has a cut between the points $z = \eta/2$ and $z = -\eta/2$. Therefore it is defined differently in the vicinities of these points

$$\mathcal{R}_n^\kappa(\lambda, z|\{\lambda\}, \{z\}) = \begin{cases} \rho(\lambda, z), & z \sim \eta/2; \\ -\kappa^{-1} \rho(\lambda, z + \eta) \cdot \prod_{b=1}^n \frac{\sinh(z - \lambda_b + \eta) \sinh(z_b - z + \eta)}{\sinh(\lambda_b - z + \eta) \sinh(z - z_b + \eta)}, & z \sim -\eta/2. \end{cases} \quad (7.4)$$

where $\rho(\lambda, z)$ is the inhomogeneous spectral density of the ground state.

Observe that the time and the distance dependence of the generating function (7.2) is associated to the bare energy $E(\lambda)$ and momentum $p(\lambda)$, making the equation (7.2) very suggestive. In the limit $t = 0$ the integrand in (7.2) is a holomorphic function in the vicinity of $-\eta/2$, therefore the integrals over $\Gamma\{-\eta/2\}$ vanish, and we arrive at the time-independent representation (5.5).

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A Solutions of the twisted Bethe equations

In this Appendix we prove Lemmas 4.1, 4.2 using the results of [44]. Consider the system of equations (4.16)

$$\mathcal{Y}_\kappa(z_j|\{z\}) = 0, \quad j = 1, \dots, N. \quad (\text{A.1})$$

Without loss of generality we can identify two solutions of this system $\{z\}$ and $\{z'\}$ if they are equal modulo $i\pi$.

It was proved in [44] that

Theorem A.1. [44] *Let κ and inhomogeneities $\{\xi\}$ be generic. Then*

- a) All admissible off-diagonal solutions of the system (A.1) are non-degenerated.*
- b) The set of the states (3.2) corresponding to the admissible off-diagonal solutions form a basis in the subspace $\mathcal{H}^{(M/2-N)}$.*

The strategy of [44] was to consider the limit of the system (A.1) at $\kappa \rightarrow 0$ and then to deform the solutions $\{z(0)\}$ to the case $\kappa \neq 0$. Then the statement a) follows from the implicit function theorem. Indeed, the solutions of the system (A.1) at $\kappa = 0$ are evident. In particular all admissible solutions have form $z_j(0) = \xi_{p_j} - \eta$. One can easily check that the Jacobian matrix $\partial \mathcal{Y}_\kappa(z_j|\{z\})/\partial z_k$ at $\kappa = 0$ for admissible solutions is a diagonal matrix with non vanishing entries. Hence, in this case the solution $\{z(\kappa)\}$ is a holomorphic deformation of the solution $\{z(0)\}$, and therefore for $|\kappa|$ small enough all admissible solutions are in the vicinities of $\{\xi - \eta\}$. On the other hand it is clear that any admissible solution at $\kappa \neq 1$ is separated from the admissible solutions at $\kappa = 1$. Thus, admissible solutions are separated from the points $\{\xi\}$ and $\{\lambda\}$. This proves Lemma 4.1.

In order to prove Lemma 4.2 we use the properties of the matrix $\Omega_\kappa(\{\lambda\}, \{\mu\}|\{\nu\})$ (3.10) formulated in the following two lemmas. Let the parameters $\lambda_1, \dots, \lambda_n$ and μ_1, \dots, μ_n be generic complex numbers, κ be arbitrary complex.

Lemma A.1. *Suppose there exist $\mu_a, \mu_b \in \{\mu\}$, such that $\mu_a = \xi_p$, $\mu_b = \xi_p - \eta$, where ξ_p is one of the inhomogeneity parameters. Then $\det_n \Omega_\kappa(\{\lambda\}, \{\mu\}|\{\lambda\}) = 0$.*

Proof. We have $d(\mu_a) = a(\mu_b) = 0$ and $t(\mu_b, \lambda_j) = t(\lambda_j, \mu_a)$. Thus, the columns $(\Omega_\kappa)_{ja}$ and $(\Omega_\kappa)_{jb}$ are proportional to each other, hence $\det_n \Omega_\kappa = 0$. \square

Lemma A.2. *Let $q^2 = e^{2\eta}$ be a root of unity, i.e. $\eta = i\pi Q/P$, where $Q < P$ are positive integers. Suppose there exists $\{\lambda_{a_1}, \dots, \lambda_{a_P}\} \subset \{\lambda\}$, such that*

$$\sinh(\lambda_{a_{j+1}} - \lambda_{a_j} + \eta) = 0, \quad \text{with} \quad \lambda_{a_{P+1}} \equiv \lambda_{a_1}. \quad (\text{A.2})$$

Then $\det_n \Omega_\kappa(\{\lambda\}, \{\mu\}|\{\lambda\}) = 0$.

Proof. It is easy to see that

$$\sum_{j=1}^P t(\mu, \lambda_{a_j}) = \sum_{j=1}^P t(\lambda_{a_j}, \mu) = 0, \quad (\text{A.3})$$

since both these sums are $i\pi$ -periodical holomorphic functions of μ vanishing at $\mu \rightarrow \pm\infty$. Hence the matrix $\Omega_\kappa(\{\lambda\}, \{\mu\}, \{\lambda\})$ contains linearly depended lines and its determinant vanishes. \square

Note that if the set $\{\lambda\}$ contains several subsets of the type (A.2), then the order of the zero of $\det_n \Omega_\kappa(\{\lambda\}, \{\mu\}|\{\lambda\})$ is not less than the number of such subsets.

If q^2 is not root of unity, then unadmissible solutions of the system (A.1) contain a pair $z_a = \xi_p$ and $z_b = \xi_p - \eta$. Off-diagonal unadmissible solutions are non-degenerated at $\kappa = 0$ [44]. Hence, due to Lemma A.1 they do not give a contribution to the integrals (4.17), (4.15). The diagonal solutions $\{z(\kappa)\}$ of (A.1) obtained as a deformation of diagonal solutions $\{z(0)\}$ at $\kappa = 0$ preserve their multiplicity [44]. Hence, they also do not give any contribution to the integrals (4.17), (4.15) due to vanishing of $\det_N \Omega_\kappa$ and $\det_N \Omega$ as soon as two or more z coincide.

If q^2 is a root of unity, then unadmissible solutions may contain ‘strings’ $\{z_{a_j}\}$ satisfying the condition (A.2). Such solutions are not isolated since the value of z_{a_1} is not fixed. We have seen however that due to Lemma A.2 the determinant of the matrix $\Omega_\kappa(\{z\}, \{\lambda\}|\{z\})$ vanishes exactly on these string type solutions. Hence, these unadmissible solutions of the system (A.1) do not contribute to the integrals (4.17), (4.15).

Thus, the only solutions of the system (A.1) which can give non-vanishing contribution to the integrals (4.17), (4.15) are admissible off-diagonal solutions, which are in the vicinities of $\{\xi - \eta\}$ for $|\kappa|$ small enough. This proves Lemma 4.2.

B The σ^z form factor

The explicit formulas for form factors of local spin operators in the finite XXZ chain were obtained in [25]. Here we propose slightly modified method to derive the form factor of the operator σ^z .

Consider a matrix element of the operator $Q_{1,m}^\kappa$ between an eigenstate $|\psi(\{\lambda\})\rangle$ of the transfer matrix $\mathcal{T}(\nu)$ (for example, ground state) and an eigenstate $\langle\psi_\kappa(\{\mu\})|$ of the operator $\mathcal{T}_\kappa(\nu)$. Using (4.5) we immediately obtain

$$\langle\psi_\kappa(\{\mu(\kappa)\})|Q_{1,m}^\kappa|\psi(\{\lambda\})\rangle = \prod_{a=1}^m \frac{\tau_\kappa(\xi_a|\{\mu(\kappa)\})}{\tau(\xi_a|\{\lambda\})} \cdot \langle\psi_\kappa(\{\mu(\kappa)\})|\psi(\{\lambda\})\rangle. \quad (\text{B.1})$$

Here we have indicated explicitly that the parameters $\{\mu\}$ depend on κ , for they are solutions of the system (3.6). On the other hand it is clear that

$$\left. \frac{\partial}{\partial \kappa} \langle\psi_\kappa(\{\mu(\kappa)\})|[Q_{1,m+1}^\kappa - Q_{1,m}^\kappa]|\psi(\{\lambda\})\rangle \right|_{\kappa=1} = \frac{1}{2} \langle\psi(\{\mu(1)\})|(1 - \sigma_{m+1}^z)|\psi(\{\lambda\})\rangle. \quad (\text{B.2})$$

Thus, we have

$$\begin{aligned} \frac{1}{2} \langle \psi(\{\mu(1)\}) | (1 - \sigma_{m+1}^z) | \psi(\{\lambda\}) \rangle &= \frac{\partial}{\partial \kappa} \prod_{a=1}^m \frac{\tau_\kappa(\xi_a | \{\mu(\kappa)\})}{\tau(\xi_a | \{\lambda\})} \cdot \left(\frac{\tau_\kappa(\xi_{m+1} | \{\mu(\kappa)\})}{\tau(\xi_{m+1} | \{\lambda\})} - 1 \right) \\ &\times \frac{\prod_{a=1}^N d(\mu_a(\kappa))}{\prod_{a>b}^N \sinh(\lambda_a - \lambda_b) \sinh(\mu_b(\kappa) - \mu_a(\kappa))} \cdot \det_N \Omega_\kappa(\{\mu(\kappa)\}, \{\lambda\} | \{\mu(\kappa)\}) \Big|_{\kappa=1}. \end{aligned} \quad (\text{B.3})$$

In order to evaluate explicitly the derivative over κ in (B.3) one should distinguish two cases: $\{\mu(1)\} = \{\lambda\}$ and $\{\mu(1)\} \neq \{\lambda\}$. In the first case $\tau_\kappa(\xi_{m+1} | \{\mu(\kappa)\}) \rightarrow \tau(\xi_{m+1} | \{\lambda\})$ as $\kappa \rightarrow 1$, therefore

$$\frac{\langle \psi(\{\lambda\}) | (1 - \sigma_{m+1}^z) | \psi(\{\lambda\}) \rangle}{\langle \psi(\{\lambda\}) | \psi(\{\lambda\}) \rangle} = 2 \frac{\partial}{\partial \kappa} \left(\frac{\tau_\kappa(\xi_{m+1} | \{\mu(\kappa)\})}{\tau(\xi_{m+1} | \{\lambda\})} - 1 \right) \Big|_{\kappa=1} \quad (\text{B.4})$$

$$= -2 \sum_{k=1}^N \frac{d\mu_k(\kappa)}{d\kappa} \Big|_{\kappa=1} \cdot t(\lambda_k, \xi_{m+1}). \quad (\text{B.5})$$

The derivatives $d\mu_k(\kappa)/d\kappa$ can be found from (3.4) via

$$\sum_{k=1}^N \frac{\partial \mathcal{Y}_\kappa(\mu_j | \{\mu\})}{\partial \mu_k} \cdot \frac{d\mu_k(\kappa)}{d\kappa} \Big|_{\kappa=1} + d(\mu_j) \prod_{a=1}^N \sinh(\mu_a - \mu_j - \eta) = 0. \quad (\text{B.6})$$

In the second case we can compute explicitly the derivative of $\det \Omega_\kappa$. Indeed, consider the N -dimensional vector-column v with the components

$$v_k = \prod_{a=1}^N \sinh(\mu_k(\kappa) - \lambda_a) \prod_{\substack{a=1 \\ a \neq k}}^N \sinh^{-1}(\mu_k(\kappa) - \mu_a(\kappa)). \quad (\text{B.7})$$

If $\{\mu(1)\} \neq \{\lambda\}$, then this vector at $\kappa = 1$ has at least one non-zero component, say $v_N \neq 0$. Then multiplying the k -th column of $(\Omega_\kappa)_{jk}$ by v_k/v_N and adding the first $(N-1)$ columns to the last one, we obtain

$$(\Omega_\kappa)_{jN} + \sum_{k=1}^{N-1} \frac{v_k}{v_N} (\Omega_\kappa)_{jk} = \frac{1}{v_N} \mathcal{Y}_\kappa(\lambda_j | \{\lambda\}) = \frac{1-\kappa}{v_N} \cdot a(\lambda_j) \prod_{a=1}^N \sinh(\lambda_a - \lambda_j + \eta). \quad (\text{B.8})$$

(see Appendix B of [31] for the proof). Thus, the last column is proportional to $1 - \kappa$, hence, taking the derivative over κ of $\det \Omega_\kappa$ one has to differentiate only this column. This gives

$$\frac{\partial}{\partial \kappa} \det_N \Omega_\kappa(\{\mu(\kappa)\}, \{\lambda\} | \{\mu(\kappa)\}) \Big|_{\kappa=1} = \det_N \tilde{\Omega}(\{\mu(1)\}, \{\lambda\} | \{\mu(1)\}), \quad (\text{B.9})$$

where

$$(\tilde{\Omega})_{jk}(\{\mu\}, \{\lambda\} | \{\mu\}) = \begin{cases} (\Omega_\kappa)_{jk}(\{\mu\}, \{\lambda\} | \{\mu\}) \Big|_{\kappa=1}, & k = 1, \dots, N-1, \\ -\frac{1}{v_N} \cdot a(\lambda_j) \prod_{a=1}^N \sinh(\lambda_a - \lambda_j + \eta), & k = N. \end{cases}$$

Thus, for $\{\mu\} \neq \{\lambda\}$ we obtain

$$\begin{aligned} \langle \psi(\{\mu\}) | \sigma_{m+1}^z | \psi(\{\lambda\}) \rangle &= 2 \prod_{a=1}^m \frac{\tau_\kappa(\xi_a | \{\mu\})}{\tau(\xi_a | \{\lambda\})} \cdot \left(1 - \frac{\tau_\kappa(\xi_{m+1} | \{\mu\})}{\tau(\xi_{m+1} | \{\lambda\})} \right) \\ &\times \frac{\prod_{a=1}^N d(\mu_a)}{\prod_{a>b}^N \sinh(\lambda_a - \lambda_b) \sinh(\mu_b - \mu_a)} \cdot \det_N \tilde{\Omega}(\{\mu\}, \{\lambda\} | \{\mu\}). \quad (\text{B.10}) \end{aligned}$$

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